

Quantum correlations in the collective spin systems

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Quantum and classical pairwise correlations in two typical collective spin systems, i.e. the Dicke model and the Lipkin-Meshkov-Glick model are discussed. In the thermodynamical limit, analytical expressions of these correlations are derived. Within the effective technique proposed previously, theses correlations for finite size system can be also obtained numerically up to very large size. The scaling behavior for the quantum discord and its first derivative are analyzed. The comparison of the quantum discord and the concurrence are performed and essentially different behaviors are observed.

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I. INTRODUCTION

Correlation, as a fundamental feature, has been extensively investigated in many-body physics and the quantum information science [1, 2]. They indeed unravel the key physical properties and characterize the remarkable phenomena of the critical systems, such as quantum phase transition (QPT) [3]. QPTs reveal the qualitative change of the quantum systems resulted from the energy level crossings at zero temperature.

Both the quantum and classical sources contribute to the correlation [4]. As one kind of the quantum correlation, the entanglement is widely applied as the characteristic trait to detect the quantum correlation [5]. Specifically, the entanglement successfully identifies the critical behaviors of the QPTs [6–11]. However, the entanglement may fail to capture the existence of the quantum correlation in some mixed separate states, in which the entanglement is considered not a good measure [12, 13]. While a new kind of the quantum correlation, quantum discord (QD), provides the alternative route for measurement, which is present even under separable states [12]. The definition of the QD can be interpreted as the difference of the total quantum information of the two subsystems A and B before and after the local operation on the one of them. The QD has been proved as a good measure of the non-classical correlations beyond entanglement. Furthermore, the QD has been indicated as the source to speed up the quantum computations [14, 15].

Much attention has been recently paid to apply the QD to quantify the critical properties of spin chain systems, especially the QPTs [16–22]. Specifically, the QPTs of one dimensional XXZ model and transverse field Ising model have been analytically characterized by the QD at zero temperature [16, 17] and the finite temperature [18, 19]. The critical properties of the half spin XY chain

have been exhibited [20]. The QD has been also studied in extended XY models [21, 22]. chains. Apart from spin systems, the QD has been used to characterize the QPTs in the correlated electron systems [23] and a topological QPT in the Castelnovo-Chamon model[24].

The Dicke model[25] and Lipkin-Meshkov-Glick (LMG) model[26] are two well known quantum collective spin ones. The Dicke model describes the interaction of N two-level atoms with a single bosonic mode and exhibits QPT at the critical atom-cavity coupling. It can be realized experimentally in a Bose-Einstein condensate in an optical cavity[27]. LMG model was originally introduced in nuclear physics, but now has found applications in other fields[28–30]. It describes N mutually interacting spins in a transverse magnetic field, and also undergoes QPT and shares the same universality with the Dicke model. Except preliminarily approximate results for the QD in the LMG model [17] in the thermodynamical limit, the QD in these collective spin systems has not been well studied to date.

In the present paper we study the QD and the classical correlation of the Dicke model and the LMG model both in the thermodynamic limit and for finite size systems. The critical behavior related to the QD and its first derivative are analyzed. The paper is outlined as follows. In Sec. II we review the definition of the QD and the classical correlation, and then apply them to the spin collective models. In Sec. III we analyze the QD, the counterpart classical correlation, and the entanglement in detail of the Dicke model, and discussions are also presented. In Sec. IV, these features are studied in LMG. Finally, we summarize our work in Sec. V.

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II. GENERAL FORMALISM FOR QUANTUM AND CLASSICAL CORRELATIONS IN THE COLLECTIVE SPIN MODELS

The mutual information to describe the total correlation of the sub-systems \mathcal{A} and \mathcal{B} can be written as

$$\mathcal{I}(\mathcal{A}; \mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B}) - H(\mathcal{A}, \mathcal{B}), \quad (1)$$

where $H(a) = -\sum_k p_{a=k} \log p_{a=k}$ ($a = \mathcal{A}, \mathcal{B}$) with $p_{a=k}$ the probability of the realization k for the sub-system a . The joint entropy of \mathcal{A} and \mathcal{B} is denoted as $H(\mathcal{A}, \mathcal{B}) = -\sum_{j,k} p_{\mathcal{A}=j, \mathcal{B}=k} \log p_{\mathcal{A}=j, \mathcal{B}=k}$ with $p_{\mathcal{A}=j, \mathcal{B}=k}$ the joint probability of the sub-systems \mathcal{A} and \mathcal{B} , realized by j and k . By using the Bayes rule $p_{\mathcal{A}|\mathcal{B}=k} = p_{\mathcal{A}, \mathcal{B}=k} / p_{\mathcal{B}=k}$, the classical mutual information can also be rewritten into the equivalent expression

$$\mathcal{J}(\mathcal{A}; \mathcal{B}) = H(\mathcal{B}) - H(\mathcal{A}|\mathcal{B}), \quad (2)$$

where $H(\mathcal{A}|\mathcal{B}) = -\sum_{j,k} p_{\mathcal{A}=j|\mathcal{B}=k} \log p_{\mathcal{A}=j|\mathcal{B}=k}$ is the conditional entropy of the \mathcal{A} and \mathcal{B} . $p_{\mathcal{A}=j|\mathcal{B}=k}$ is the corresponding conditional probability.

For the quantum systems, the two definitions of the mutual information in Eqs. (1) and (2) are replaced by the Von Neumann entropy. The joint entropy is shown as $H(\mathcal{A}, \mathcal{B}) = -\text{Tr}_{\mathcal{A}, \mathcal{B}} \{\rho_{\mathcal{A}, \mathcal{B}} \log \rho_{\mathcal{A}, \mathcal{B}}\}$, where $\rho_{\mathcal{A}, \mathcal{B}}$ is the density matrix of the total system. Similarly, the Von Neumann entropy for the sub-system is $H(\mathcal{A}) = -\text{Tr}_{\mathcal{A}} \{\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\}$, with $\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}} \{\rho_{\mathcal{A}, \mathcal{B}}\}$. The quantum conditional entropy quantifies the missing information of \mathcal{A} after selecting the state of \mathcal{B} . It can be carried out from the conditional density matrix

$$\rho_{\mathcal{A}|\Pi_k^{\mathcal{B}}} = \Pi_k^{\mathcal{B}} \rho_{\mathcal{A}, \mathcal{B}} \Pi_k^{\mathcal{B}} / p_k, \quad (3)$$

where $\Pi_k^{\mathcal{B}}$ is the projector onto the state k of the sub-system \mathcal{B} and the probability is $p_k = \text{Tr}_{\mathcal{A}, \mathcal{B}} \{\Pi_k^{\mathcal{B}} \rho_{\mathcal{A}, \mathcal{B}}\}$. Usually, these two mutual information become different, which leads to the emergence of the quantum discord $\mathcal{D}(\mathcal{A} : \mathcal{B}) = \mathcal{I}(\mathcal{A}; \mathcal{B}) - \mathcal{J}(\mathcal{A}; \mathcal{B})$, described in Refs. [12, 31]. In the quantum measurements, the quantum discord between \mathcal{A} and \mathcal{B} reads

$$\mathcal{D}(\mathcal{A} : \mathcal{B}) = \min_{\{\Pi_k^{\mathcal{B}}\}} \{H(\mathcal{A}) - H(\mathcal{A}, \mathcal{B}) + H(\mathcal{A}|\{\Pi_k^{\mathcal{B}}\})\}. \quad (4)$$

Then the classical correlation is described as

$$\mathcal{C}(\mathcal{A} : \mathcal{B}) = \max_{\{\Pi_k^{\mathcal{B}}\}} \{H(\mathcal{A}) - H(\mathcal{A}|\{\Pi_k^{\mathcal{B}}\})\}. \quad (5)$$

Here, we apply the quantum discord to the collective spin systems, where the Dicke and LMG models are two typical examples. Since the quantum discord is adopted to describe the nonlocal quantum correlation, it is necessary to derive the pairwise density matrix. In this paper, we only study the pairwise correlations. For the Dicke model and LMG model, the reduced pairwise matrix in

the standard basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ (with $\sigma_z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$) [32], can be derived as

$$\rho = \begin{pmatrix} v_+ & x_+^* & x_+^* & u^* \\ x_+ & w & y & x_-^* \\ x_+ & y & w & x_-^* \\ u & x_- & x_- & v_- \end{pmatrix}. \quad (6)$$

The detailed expressions for these elements are

$$\begin{aligned} v_{\pm} &= \frac{N^2 - 2N + 4\langle J_z^2 \rangle \pm 4(N-1)\langle J_z \rangle}{4N(N-1)}, \\ x_{\pm} &= \frac{(N-1)\langle J_{\pm} \rangle \pm \langle [J_{\pm}, J_z]_{\pm} \rangle}{2N(N-1)}, \\ w &= \frac{N^2 - 4\langle J_z^2 \rangle}{4N(N-1)}, y = \frac{\langle J_x^2 + J_y^2 \rangle - N/2}{N(N-1)}, \\ u &= \frac{\langle J_+^2 \rangle}{N(N-1)}, \end{aligned} \quad (7)$$

where $[A, B]_{\pm} = AB + BA$. $w = y$, for $\sum_{\alpha=x,y,z} J_{\alpha}^2 = J^2 = \frac{N}{2}(\frac{N}{2} + 1)$. For the symmetric states with parity conservation, we find $x_{\pm} = 0$ from Refs. [11, 32]. Hence the pairwise reduced density matrix is shown in X form.

From the definition in Eq. (4), the quantum discord \mathcal{D} and the classical correlation \mathcal{C} can be obtained over the reduced sub-system von Neumann entropy $H_{\mathcal{A}}$, the joint entropy $H_{\mathcal{A}, \mathcal{B}}$, and the conditional entropy $H_{\mathcal{A}|\Pi_k^{\mathcal{B}}}$, which are analyzed in detail in Appendix A.

III. QUANTUM CORRELATION OF THE DICKE MODEL

We study the QD in the Dicke model. The Dicke Hamiltonian can be written in terms of the collective momentum form [33, 34]

$$H_{\text{Dicke}} = \omega a^{\dagger} a + \Delta J_z + \frac{2\lambda}{\sqrt{N}} (a^{\dagger} + a) J_x. \quad (8)$$

where a^{\dagger} and a are the bosonic annihilation and creation operators of the single-mode cavity, Δ and ω are the transition frequency of the qubit and the frequency of the single bosonic mode, λ is the coupling constant. J_x and J_z are the collective spin operators. It is well known that this model undergoes a second order quantum phase transition from the normal phase to the super-radiant phase, separated by the critical point $\lambda_c = \sqrt{\omega\Delta}/2$.

We first apply the Holstein-Primakoff transformation to change the collective angular operators to the boson operators $b(b^{\dagger})$ by $J_+ = b^{\dagger}\sqrt{N-b^{\dagger}b}$, $J_- = \sqrt{N-b^{\dagger}b}b$, and $J_z = b^{\dagger}b - N/2$, where $[b, b^{\dagger}] = 1$ [33]. Then the displacements of the boson operators are introduced to depict the behaviors of super-radiation phase as $c^{\dagger} = a^{\dagger} + \sqrt{N}\alpha$ and $d^{\dagger} = b^{\dagger} - \sqrt{N}\beta$. By using large N expansions of H_{Dicke} with respect to the new boson operators c^{\dagger} and d^{\dagger} up to the $1/N$, we obtain the ground state energy as

$$\frac{E_G(\alpha, \beta)}{N} = \omega\alpha^2 - 4\lambda\alpha\beta\sqrt{1-\beta^2} + \Delta(\beta^2 - 1). \quad (9)$$

Minimizing the ground state energy gives

$$\begin{aligned}\omega\alpha - 2\lambda\beta\sqrt{1-\beta^2} &= 0 \\ 2\alpha\lambda\sqrt{1-\beta^2} - \frac{2\alpha\lambda\beta^2}{\sqrt{1-\beta^2}} - \beta\Delta &= 0.\end{aligned}\quad (10)$$

then we have

$$\begin{aligned}\beta^2 &= \max\{0, \frac{1}{2}(1 - \lambda_c^2/\lambda^2)\}, \\ \alpha &= \frac{2\lambda}{\omega}\beta\sqrt{1-\beta^2}.\end{aligned}\quad (11)$$

where the transition point $\lambda_c = \sqrt{\omega\Delta}/2$. Next we can derive the matrix elements of the pairwise reduced density in Eq. (6) up to $O(1)$

$$\begin{aligned}v_+ &= \beta^4, \quad v_- = (1 - \beta^2)^2, \\ w &= y = \beta^2(1 - \beta^2), \quad u = \beta^2(1 - \beta^2).\end{aligned}\quad (12)$$

The von Neumann entropy of the subsystem \mathcal{A} and \mathcal{B} are thus given by

$$H(\mathcal{A}(\mathcal{B})) = -\beta^2 \log \beta^2 - (1 - \beta^2) \log(1 - \beta^2), \quad (13)$$

$$\begin{aligned}H(\mathcal{A}, \mathcal{B}) &= -[2\beta^2(1 - \beta^2)] \log[2\beta^2(1 - \beta^2)] \\ &\quad -[(\beta^4 + (1 - \beta^2)^2)] \log[\beta^4 + (1 - \beta^2)^2].\end{aligned}\quad (14)$$

Note that the quantum mutual information is obtained by $\mathcal{I}(\rho_{\mathcal{A}, \mathcal{B}}) = H(\mathcal{A}) + H(\mathcal{B}) - H(\mathcal{A}, \mathcal{B})$, so we can derive the Eq. A4 and Eq. A5 as

$$\begin{aligned}(X_{k,+} + X_{k,-}) &= \eta_k(\theta), \\ (X_{k,+} - X_{k,-}) &= (2\beta^2 - 1)\eta_k(\theta), \\ |Y_k|^2 &= |Y|^2 = \beta^4(1 - \beta^2)^2 \sin^2 2\theta \cos^2 \phi, \\ p_k &= \eta_k(\theta),\end{aligned}\quad (15)$$

with $k = 1, 2$.

$$\eta_k(\theta) = \frac{1}{2}[1 + (-1)^{k-1}(2\beta^2 - 1)\cos 2\theta].$$

Therefore the eigenvalues of the conditional density matrix in Eq. A3 are obtained as

$$\begin{aligned}\lambda_{\pm}^k(\theta, \phi) &= \frac{\eta_k(\theta) \pm [(2\beta^2 - 1)^2 \eta_k^2(\theta) + 4|Y|^2]^{1/2}}{2p_k} \\ &= \frac{\eta_k(\theta)}{2p_k} \{1 \pm [(2\beta^2 - 1)^2 + \frac{4|Y|^2}{\eta_k^2(\theta)}]^{1/2}\}.\end{aligned}\quad (16)$$

By defining $x_k(\theta, \phi) = [(2\beta^2 - 1)^2 + \frac{4|Y|^2}{\eta_k^2(\theta)}]^{1/2}$ ($0 < x_k < 1$), we simplify the $\lambda_{\pm}^k(\theta, \phi) = \frac{\eta_k(\theta)}{2p_k} [1 \pm x_k(\theta, \phi)]$. The condi-

tional von Neumann entropy is described as

$$\begin{aligned}H(\mathcal{A}|\Pi^{\mathcal{B}})(\theta, \phi) &= \sum_{k=1,2} \{\eta_k(\theta) \log \eta_k(\theta) \\ &\quad - [\frac{\eta_k(\theta)}{2}(1 + x_k(\theta, \phi))] \log [\frac{\eta_k(\theta)}{2}(1 + x_k(\theta, \phi))] \\ &\quad - [\frac{\eta_k(\theta)}{2}(1 - x_k(\theta, \phi))] \log [\frac{\eta_k(\theta)}{2}(1 - x_k(\theta, \phi))]\} \\ &= \log 2 - \sum_{k=1,2} \frac{\eta_k(\theta)}{2} [(1 + x_k(\theta, \phi)) \log(1 + x_k(\theta, \phi)) \\ &\quad + (1 - x_k(\theta, \phi)) \log(1 - x_k(\theta, \phi))],\end{aligned}\quad (17)$$

Since $F(x) = -[(1 + x) \log(1 + x) + (1 - x) \log(1 - x)]$ is the monotonically decreasing function, ϕ is selected to 0 to maximize $x_k(\theta, \phi)$. Furthermore, we find $\theta = \pi/4$ to minimize the conditional von Neumann entropy, with

$$M = x_k(\pi/4, 0) = [(2\beta^2 - 1)^2 + 16\beta^4(1 - \beta^2)^2]^{1/2}. \quad (18)$$

As a result, the conditional von Neumann entropy is shown as

$$\begin{aligned}H(\mathcal{A}|\Pi^{\mathcal{B}}) &= \log(2) - \frac{1}{2}[(1 + M) \log(1 + M) \\ &\quad + (1 - M) \log(1 - M)].\end{aligned}\quad (19)$$

Combining Eqs. (13), (14), and (19), we derive the quantum discord as

$$\begin{aligned}\mathcal{D} &= -\beta^2 \log \beta^2 - (1 - \beta^2) \log(1 - \beta^2) \\ &\quad + [2\beta^2(1 - \beta^2)] \log[2\beta^2(1 - \beta^2)] \\ &\quad + [\beta^4 + (1 - \beta^2)^2] \log[\beta^4 + (1 - \beta^2)^2] + \log 2 \\ &\quad - \frac{1}{2}[(1 + M) \log(1 + M) + (1 - M) \log(1 - M)].\end{aligned}\quad (20)$$

While the classical correlation is

$$\begin{aligned}\mathcal{C} &= -\beta^2 \log \beta^2 - (1 - \beta^2) \log(1 - \beta^2) - \log 2 \\ &\quad + \frac{1}{2}[(1 + M) \log(1 + M) + (1 - M) \log(1 - M)].\end{aligned}\quad (21)$$

We next investigate the quantum and classical correlation in the finite size Dicke model. Two of the present authors and collaborators have proposed a numerically exact technique to the Dicke model [34] up to very huge size by using the basis of extended coherent states. This effective approach has been confirmed recently by comparing with the results in terms of basis of the Fock states [35]. It was demonstrated [35] that it is very difficult to obtain convergent results for large number of atoms based on usual basis of the Fock states [36].

In the numerically exact approach [34], the wave function can be expressed in terms of the basis $\{|\varphi_n\rangle_b \otimes |j, n\rangle\}$ where $|j, n\rangle$ is the Dicke state with $j = N/2$ and $|\varphi_n\rangle_b$ is the bosonic extended coherent state

$$|\varphi_n\rangle_b = \sum_{k=0}^{N_{tr}} c_{n,k} \frac{1}{\sqrt{k!}} (a^\dagger + g_n)^k e^{-g_n a^\dagger - g_n^2/2} |0\rangle_a, \quad (22)$$

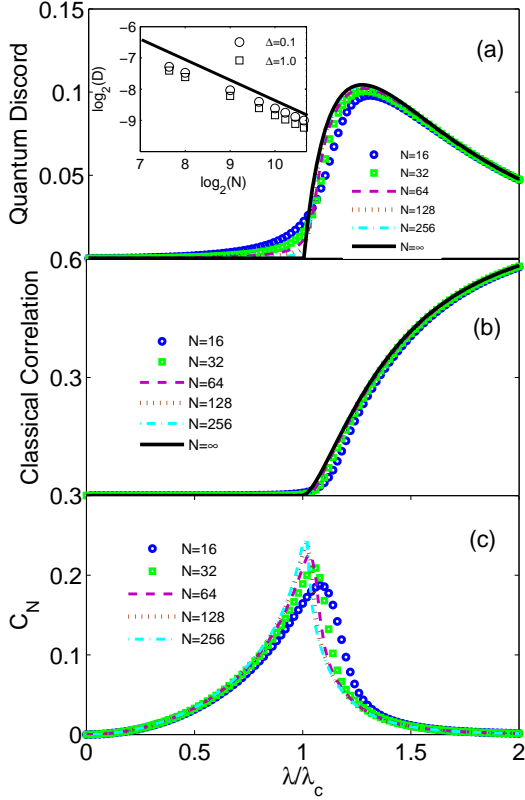


FIG. 1: (Color online) (a) Quantum discord, (b) classical correlation, and (c) concurrence as a function of the coupling constant in the Dicke model with different sizes for $\Delta = 1.0$ and $\omega = 1.0$. Inset in (a) is the finite size scaling of the quantum discord at the critical point λ_c , and the solid line scales as $N^{-2/3}$.

where $g_n = 2\lambda n/\omega\sqrt{N}$, N_{tr} is the truncated bosonic number in the space of the new operator $A_n = a + g_n$, $|0\rangle_a$ is the vacuum as $a|0\rangle_a = 0$, and the coefficient $c_{n,k}$ can be determined through the exact Lanczos diagonalization. Then, we can derive the elements of pairwise density matrix in Eq. (6). Without loss of generality, we mainly focus on the resonant case $\Delta = \omega$ in the following.

As is well known that the second-order quantum phase transition occurs in the thermodynamic limit. The quantum correlation is deeply related with the quantum phase transition. The reminiscent nature of the quantum phase transition in the large size system should be very interesting. Therefore, we study the size dependence of the QD and classical correlation in the Dicke model. In Fig. 1 (a) and (b), we displayed the quantum correlation and classical correlation as a function of the atom-cavity coupling constant for several system sizes. The results in the thermodynamic limit are also listed.

It is shown that both the quantum correlation and classical correlation are quite small in the normal phase ($\lambda < \lambda_c$). They tend to zero with the increase of the atomic number, agreeing well with the results in thermodynamic limit, which is exactly zero. In the ther-

modynamic limit, there is no excitation of the system in the normal phase, so correlation between two arbitrary atoms, which can only be mediated by the photons, should be absent. While in the super-radiant phase ($\lambda > \lambda_c$), the QD shows non-monotonous behavior as the coupling increases in Fig. 1 (a). The maximum of the QD is away from the critical point, even in the thermodynamic limit, locating at $1.28\lambda_c$. It is different from one dimensional XXZ model [16], where the maximum of the QD is right at the critical point.

It is very interesting to observe a power law scaling $\mathcal{D}|_{\lambda=\lambda_c} \propto N^{-\mu}$ for different detunings, as shown in the inset of Fig. 1 (a). The asymptotic slope in the log-log scale in the large size atomic systems for different detunings suggests the same exponent $\mu = 2/3$. To the best of our knowledge, such a finite size scaling for the QD itself has never been reported in other critical systems.

As the coupling strength increase further, the QD becomes smaller. In the strong coupling limit, the QD finally decreases to zero, which means the non-local quantum correlation of the system disappears. It can be also seen in the Eq. (20) that $\beta^2 = 1/2$ as $\lambda \rightarrow \infty$, so $\mathcal{D} \rightarrow 0$.

The classical correlation in the super-radiant phase increases monotonously with the coupling strength, similar to those in the different spin chains [16, 21]. In the thermodynamical limit, $\beta^2 \rightarrow 1/2$ in the deep coupling regime, so that $H(\mathcal{A}) = H(\mathcal{B}) = \log 2$, and $H(\mathcal{A}, \mathcal{B}) = \log 2$. For the conditional von Neumann entropy, $H(\mathcal{A}|\Pi^{\mathcal{B}}) = 0$. Hence, the classical correlation saturates at $\log 2$ in the strong coupling limit.

It is well known that the concurrence and the QD are both good measures to investigate the nonclassical correlations. Hence, comparisons between these two quantities are highly desirable. The scaled concurrence has been calculated previously [36]. The concurrence in the very large system size has been calculated later by two present authors and collaborators [34]. We also collect the concurrence as a function of coupling constant for different size in Fig. 1 (c) for convenience.

Interestingly, we find that the quantum discord and the concurrence show essentially different behaviors in the whole coupling regime. First, the concurrence increases with the atomic number in the normal phase, it is always finite in the thermodynamic limit, in sharp contrast with the QD in the Dicke model. It is shown that the QD decreases with the atomic number in the normal phase and vanish exactly in the thermodynamic limit. Therefore we conclude that the concurrence is not sufficient to quantify the non-classical correlation. Second, the concurrence reaches its maximum at the critical point where the QD is still very small. Third, in the superradiant phase, the QD show the broad maximum and decay slowly, while the concurrence decreases monotonously and rapidly with the coupling constant in the strong coupling regime. E.g. for $\lambda/\lambda_c = 2.0$, the concurrence becomes negligible, and the value of QD still remains about half of the maximum, demonstrating the robustness of the QD. In this sense, one can say that

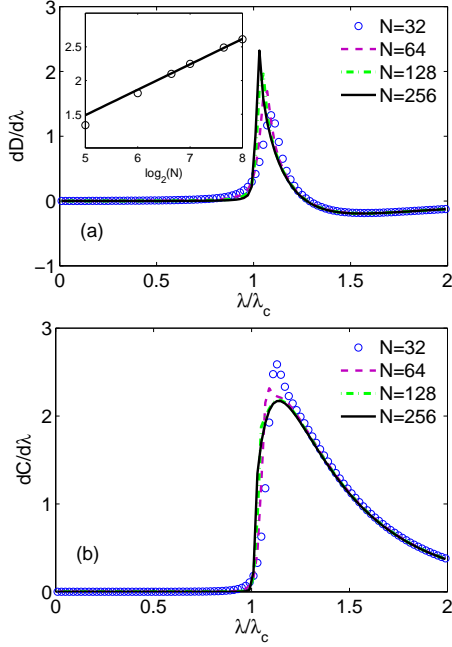


FIG. 2: (Color online) First derivative of the quantum discord (a) and the classical counterpart (b) as a function of the coupling constant in the Dicke model with different system sizes for $\Delta = 1.0$ and $\omega = 1.0$. Inset in (a) shows the finite size scaling of the maximum of the $d\mathcal{D}/d\lambda$.

QD is a pretty good measure of the quantum correlation independent of entanglement.

To quantify the quantum phase transition of the Dicke model, the first derivative of the QD and the classical correlation as a function of the coupling constant are presented in Fig. 2. The cusp-like peak of $\frac{d\mathcal{D}}{d\lambda}$ curves emerges with increase of the atomic number, indicating an non-analytical behavior. The peak position tends to the critical point with increasing N . It would provide a good tool to detect the critical point by a finite size study. In the spin chains, the first derivative of the QD also shows its peak right at the critical point, which is also non-analytical and discontinuous.

Interestingly, the logarithmic scaling of the maximum of the $d\mathcal{D}/d\lambda$ is also observed in the large size regime, which can be fitted well by $(\frac{d\mathcal{D}}{d\lambda})_{\max} = 0.377 \log_2(N) - 0.401$, demonstrated in the inset of Fig. 2. This logarithmic divergence has been also reported in Ref. [17, 22] for XY spin chain and transverse field Ising chain. The first derivative of the classical correlation $\frac{dC}{d\lambda}$ display a different behavior. It becomes more rounded around the maximum with the increase of the size.

IV. QUANTUM CORRELATION OF THE LMG MODEL

We turn to the other quantum collective spin model, LMG model. Its Hamiltonian reads [26, 29, 30]

$$H_{\text{LMG}} = -\frac{1}{2N} \sum_{i < j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - \frac{\lambda}{2} \sum_i \sigma_i^z. \quad (23)$$

where $\sigma_i(i = x, y, z)$ are Pauli spin-1/2 operators, λ is magnetic field, γ is anisotropic parameter. In the framework of the collective spin operators $J_k = \sum_i \sigma_i^k/2$ with $k = x, y, z$, and $J_{\pm} = J_x \pm iJ_y$, the model can be rewritten as

$$H_{\text{LMG}} = -\lambda J_z - \frac{1}{N} [J_x^2 + \gamma J_y^2 - N(1 + \gamma)/4]. \quad (24)$$

We focus on the case of $0 \leq \gamma < 1$ and $\lambda > 0$ where the second-order quantum phase transition can occurs.

We also first study the quantum and classical correlation in the thermodynamic limit. Based on the Holstein-Primakoff Transformation $J_z = b^\dagger b - N/2$, $J_+ = b^\dagger \sqrt{N - b^\dagger b}$, $J_- = \sqrt{N - b^\dagger b} b$, We displace the boson operator to the form of $c^\dagger = b^\dagger + \alpha$. By large N expansion of H_{LMG} , the ground state energy per spin is shown as

$$E_G/N = -[(1 - \alpha^2)\alpha^2 + \lambda(\alpha^2 - 1/2)]. \quad (25)$$

By minimizing E_G , we find

$$\alpha^2 = \min\{1, \frac{1 + \lambda}{2}\}. \quad (26)$$

Consequently, we derive $\langle J_z \rangle/N = (\alpha^2 - 1/2)$, $\langle J_z^2 \rangle/N^2 = (\alpha^2 - 1/2)^2$, and $\langle J_+^2 \rangle/N^2 = \alpha^2(1 - \alpha^2)$. Then the elements of the pairwise density matrix in Eq. (6) up to $O(1)$ are derived by

$$\begin{aligned} v_+ &= \left(\frac{1 + \lambda}{2}\right)^2, \quad v_- = \left(\frac{1 - \lambda}{2}\right)^2, \\ w &= y = \left(\frac{1 + \lambda}{2}\right)\left(\frac{1 - \lambda}{2}\right), \quad u = \left(\frac{1 + \lambda}{2}\right)\left(\frac{1 - \lambda}{2}\right). \end{aligned} \quad (27)$$

Similar to the expressions of the elements in Dicke model in Eq. (12), we obtain the corresponding relation in the LMG model

$$\begin{aligned} \beta^2 &= \frac{1 - \lambda}{2}, \\ 1 - \beta^2 &= \frac{1 + \lambda}{2}. \end{aligned} \quad (28)$$

then the elements can expressed as

$$\begin{aligned} v_+ &= (1 - \beta^2)^2, \\ v_- &= \beta^4, \\ w &= y = \beta^2(1 - \beta^2), \\ u &= \beta^2(1 - \beta^2). \end{aligned} \quad (29)$$

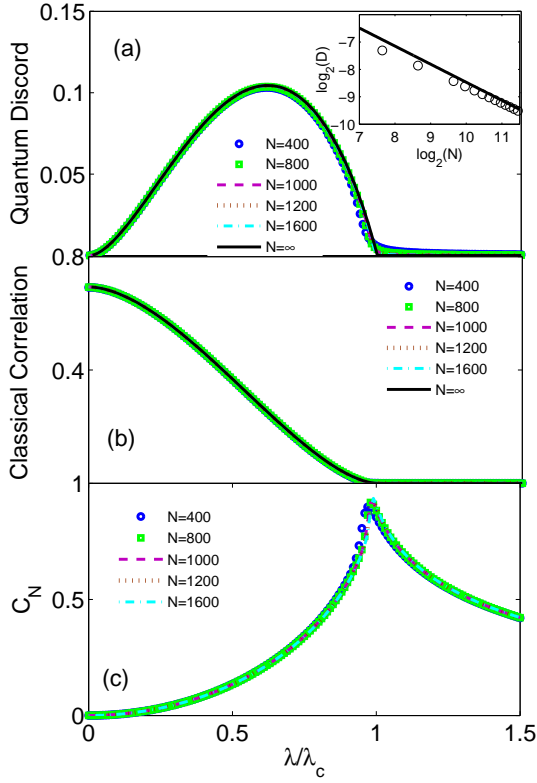


FIG. 3: (Color online) (a) Quantum discord, (b) classical correlation, and (c) concurrence as a function of the field in the LMG model with different system sizes for $\gamma = 0$. Inset in (a) is the finite size scaling of the quantum discord at the critical point λ_c , and the solid line scales as $N^{2/3}$.

The influence of the anisotropic parameter γ on the quantum correlation is checked, and we find that the QD is almost not altered with γ . Hence we mainly focus on $\gamma = 0$ in the following.

The finite size LMG model can be studied by the exact diagonalization on the basis of the collective spin operators [29]. For the ground-state, the convergent results can be obtained for very huge system. The size dependence of the QD and classical correlation is given in Fig. 3. The QD shows the non-monotonous behavior in the symmetry broken phase ($\lambda < \lambda_c$). Starting at 0, the QD then shows the single broad maximum at $\lambda = 0.62\lambda_c$, far from the critical point. While the QD is very small in the symmetry phase ($\lambda > \lambda_c$). The classical correlation decreases monotonously in the symmetry broken phase, approaching to zero at the critical point. If the magnetic field $\lambda = 0$, the classical correlation is easily obtained as $\log 2$. While in the symmetry phase, the classical correlation is rather small for finite size system and becomes zero in the thermodynamic limit.

Similar to the Dicke model, the QD of the LMG model also shows the power law scaling obviously as $\mathcal{D}_{\lambda=\lambda_c} \propto N^{-\mu}$ at the critical point, as shown in the inset of Fig. 3(a). The scaling exponent μ in the large N regime is very close to $2/3$, the same as that obtained

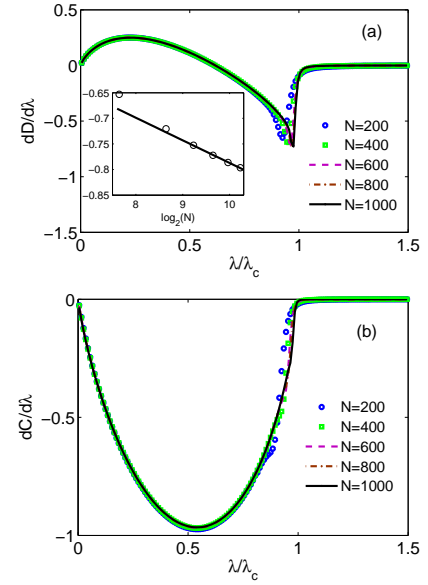


FIG. 4: (Color online) First order derivative of the quantum discord (a) and the classical counterpart (b) as a function of the field in the LMG model with different system sizes for $\gamma = 0$. Inset in (a) exhibits the finite size scaling of the minimum of the $dD/d\lambda$.

above in the Dicke model, providing a new piece of evidence of the same universality class of these two models.

The QD in the LMG model in the thermodynamic limit has been preliminarily studied in Ref. [17]. By mapping LMG model to the two bands fermion model exactly, they alternatively studied the quantum correlation and the classical counterpart of the fermions based on the corresponding density matrix. They found that both the QD and the classical counterpart decrease monotonously in the symmetry broken phase with the magnetic field, and keep zero in the symmetry broken phase. In the present paper, the pairwise density matrix is directly derived from the qubit in LMG model, which is related but not identical with that in Ref. [17]. The different definition may account for the different QD obtained in two papers. Our results for the classical correlation are consistent qualitatively with those in Ref. [17].

The comparisons of the QD and the scaled concurrence in the LMG model are also made. The scaled concurrence for large system size has been calculated by Dusuel et al[37], which was shown in their Fig. 9. For completeness, we recalculate the concurrence for much too larger system size and list them in Fig. 3(c). At the critical point, the concurrence shows the maximum, while the QD becomes 0. In the symmetry phase ($\lambda > \lambda_c$), the quantum discord are very small, implying no quantum correlation. While the concurrence remain finite and decrease gradually and monotonously with the field λ .

In order to study the critical behaviors of the LMG model, we display the first derivative of the QD and classical correlation in Fig. 4. $\frac{dD}{d\lambda}$ has a pronounced mini-

imum around the critical point. Note that the minimum position approaches the critical point, providing a convincing method to locate the critical point. This behavior strongly implies the sudden transition from the symmetry broken phase to the symmetry phase. The finite size scaling of the minimum of the $dD/d\lambda$ is also performed in inset of the Fig. 3. The logarithmic divergence fitted by $\frac{dD}{d\lambda}|_{\min} = -0.044 \log_2(N) - 0.346$ is also well exhibited in the large size regime, similar to that in the Dicke model. It should be point out that the coefficients of $\log_2(N)$ in the two models are different. Combined with the similar scaling behavior reported in spin chain [17, 22], such a logarithmic divergence of the first derivative of the QD may be universal in critical systems, further confirmations in extensive studies are however needed. The classical correlation shows the broad valley around $0.55\lambda_c$, and becomes 0 as the field parameter $\lambda \geq \lambda_c$.

V. SUMMARY

The quantum discord and its classical counterpart in the Dicke model and the LMG model have been investigated in this paper. These correlations in the thermodynamic limit have been derived analytically and in the finite size system are obtained numerically. Power law scaling behavior at the critical point is observed in both models. Such a scaling behavior has not been reported in other critical systems, as far as we know. The same scaling exponents provide a new evidence of the same universality of two models. The position of the pronounced maximum or minimum of the first derivative of the QD approaches to the critical point with the increase of the system size, which can be used to identify the quantum phase transitions. A logarithmic diverging behavior for the first derivative of the quantum discord is also observed in the two models. The coefficient of the logarithmic term is model dependent.

We also find that the quantum discord and the concurrence show essentially different behaviors in both models. In the normal phase of the Dicke model and the symmetry phase of the LMG model, the QD is zero in the thermodynamical limit and extremely small in the finite size system, while the concurrence remain finite. In the superradiant phase of the Dicke model and the symmetry broken phase of the LMG model, the QD show the broad maximum and decay slowly, while the concurrence decreases monotonously and rapidly when away from the critical point. These salient features might be used for the applications of the quantum correlation in some experimentally realized systems to the quantum information science and the quantum computing.

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Appendix A: Derivation of the quantum correlation

Considering the pairwise atom reduced density matrix in Eq. 6, the reduced density matrix $\rho_{\mathcal{A}}$ is obtained by

$$\begin{aligned}\rho_{\mathcal{A}} &= \text{Tr}_{\mathcal{B}}\{\rho_{\mathcal{A},\mathcal{B}}\} \\ &= (v_+ + w)|e\rangle_{\mathcal{A}}\langle e| + (v_- + w)|g\rangle_{\mathcal{A}}\langle g|.\end{aligned}$$

The Von Neumann entropy of the reduced system \mathcal{A} is shown as

$$H(\mathcal{A}) = -(v_+ + w)\log(v_+ + w) - (v_- + w)\log(v_- + w). \quad (\text{A1})$$

The joint von Neumann entropy can be derived from the joint pairwise matrix in Eq. 6. By solving the Eq. 6, we obtained

$$(\lambda + y - w)[(w - \lambda) + y][(\lambda - v_+)(\lambda - v_-) - |u|^2] = 0.$$

The eigenvalues $\lambda_i (i = 1, 2, 3, 4)$ can be obtained analytically. Considering $x_{\pm} = 0$, the joint entropy is demonstrated as

$$\begin{aligned}H(\mathcal{A}, \mathcal{B}) &= -(w + y)\log(w + y) - (w - y)\log(w - y) \\ &\quad - \sum_{\lambda=\lambda_{\pm}} \lambda \log \lambda,\end{aligned} \quad (\text{A2})$$

where

$$\lambda_{\pm} = \frac{1}{2}\{(v_+ + v_-) \pm [(v_+ - v_-)^2 + 4|u|^2]^{1/2}\}.$$

The conditional density $\rho_{\mathcal{A}|\Pi_k^{\mathcal{B}}}$ is measured by the projections tuned by θ and ϕ

$$\begin{aligned}|\Psi_1\rangle_{\mathcal{B}} &= \cos(\theta)|e\rangle_{\mathcal{B}} + e^{i\phi}\sin(\theta)|g\rangle_{\mathcal{B}} \\ |\Psi_2\rangle_{\mathcal{B}} &= e^{-i\phi}\sin(\theta)|e\rangle_{\mathcal{B}} - \cos(\theta)|g\rangle_{\mathcal{B}}.\end{aligned}$$

Under such projections, the conditional density matrix is shown as

$$\begin{aligned}\rho_{\mathcal{A}|\Pi_{\Psi_{\alpha}}^{\mathcal{B}}} &= |\alpha\rangle_{\mathcal{B}}\langle\alpha|\{ |e\rangle_{\mathcal{A}}\langle e|X_{\alpha,+} + |g\rangle_{\mathcal{A}}\langle g|X_{\alpha,-} \\ &\quad + |e\rangle_{\mathcal{A}}\langle g|Y_{\alpha} + |g\rangle_{\mathcal{A}}\langle e|Y_{\alpha}^*\}/p_{\alpha}.\end{aligned} \quad (\text{A3})$$

For $\alpha = 1$,

$$\begin{aligned}X_{1,+} &= v_+\cos^2(\theta) + w\sin^2(\theta), \\ X_{1,-} &= w\cos^2(\theta) + v_-\sin^2(\theta), \\ Y_1 &= \sin(\theta)\cos(\theta)[e^{i\phi}u^* + e^{-i\phi}y], \\ p_1 &= w + v_+\cos^2(\theta) + v_-\sin^2(\theta).\end{aligned} \quad (\text{A4})$$

For $\alpha = 2$,

$$\begin{aligned}X_{2,+} &= v_+\sin^2(\theta) + w\cos^2(\theta), \\ X_{2,-} &= w\sin^2(\theta) + v_-\cos^2(\theta), \\ Y_2 &= -\sin(\theta)\cos(\theta)[e^{i\phi}u^* + e^{-i\phi}y], \\ p_2 &= w + v_+\sin^2(\theta) + v_-\cos^2(\theta).\end{aligned} \quad (\text{A5})$$

Then the eigenvalues of the density matrix Eq. A3 read

$$\lambda_{\pm}^{\alpha}(\theta, \phi) = \frac{1}{2p_{\alpha}} \{ (X_{\alpha,+} + X_{\alpha,-}) \pm [(X_{\alpha,+} - X_{\alpha,-})^2 + 4|Y_{\alpha}|^2]^{1/2} \}. \quad (\text{A6})$$

Finally, the conditional Von Neumann entropy is shown as

$$H(\mathcal{A}|\{\Pi_k^{\mathcal{B}}\}) = - \sum_{\alpha=1,2} p_{\alpha} [\lambda_{+}^{\alpha}(\theta, \phi) \log \lambda_{+}^{\alpha}(\theta, \phi) + \lambda_{-}^{\alpha}(\theta, \phi) \log \lambda_{-}^{\alpha}(\theta, \phi)]. \quad (\text{A7})$$

Hence

$$\delta(\theta, \phi) = H(\mathcal{A}) - H(\mathcal{A}, \mathcal{B}) + H(\mathcal{A}|\Pi^{\mathcal{B}}). \quad (\text{A8})$$

The quantum discord can be obtained by optimizing the θ and ϕ both in the regime $[0, \pi/2]$ to minimize $\delta(\theta, \phi)$, shown as

$$\mathcal{D} = \min_{\{\theta, \phi\}} \{\delta(\theta, \phi)\}. \quad (\text{A9})$$

Consequently, the corresponding classical correlation can also be obtained by

$$\mathcal{C} = \max_{\{\theta, \phi\}} \{H(\mathcal{A}) + H(\mathcal{B}) - H(\mathcal{A}, \mathcal{B}) - \delta(\theta, \phi)\}. \quad (\text{A10})$$

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- [1] P. L. Taylor and O. Heinonen, *A Quantum Approach to Condensed Matter Physics* (Cambridge University Press, Cambridge, UK, 2002).
 - [2] M. A. Nielsen and I. L. Chuan, *Quantum Computational and Quantum information* (Cambridge University Press, Cambridge, UK 2000).
 - [3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 2001).
 - [4] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, arXiv:1112.6238 (2011).
 - [5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
 - [6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).
 - [7] S. J. Gu, H. Q. Lin, and Y. Q. Li, Phys. Rev. A **68**, 042330 (2003).
 - [8] L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004).
 - [9] S. J. Gu, S. S. Deng, Y. Q. Li, and H. Q. Lin, Phys. Rev. Lett. **93**, 086402 (2004).
 - [10] Ö. Legeza and J. Sólyom, Phys. Rev. Lett. **96**, 116401 (2006).
 - [11] J. Vidal, Phys. Rev. A **73**, 062318 (2006).
 - [12] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
 - [13] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).
 - [14] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **100**, 050502 (2008).
 - [15] B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, Phys. Rev. Lett. **101**, 200501 (2008).
 - [16] R. Dilenschneider, Phys. Rev. B **78**, 224413 (2008).
 - [17] M. S. Sarandy, Phys. Rev. A **80**, 022108 (2009).
 - [18] T. Werlang and G. Rigolin, Phys. Rev. A **81**, 044101 (2010).
 - [19] T. Werlang, C. Trippe, G. A. P. Ribeiro, and G. Rigolin, Phys. Rev. Lett. **105**, 095702 (2010).
 - [20] J. Maziero, H. C. Guzman, L. C. Céleri, M. S. Sarandy, and R. M. Serra, Phys. Rev. A **82**, 012106 (2010).
 - [21] B. Q. Liu, B. Shao, J. G. Li, J. Zou, and L. A. Wu, Phys. Rev. A **83**, 052112 (2011).
 - [22] Y. C. Li and H. Q. Lin, Phys. Rev. A **83**, 052323 (2011).
 - [23] M. Allegra, P. Giorda, and A. Montorsi, Phys. Rev. B **84**, 245133 (2011).
 - [24] Y. X. Chen and S. W. Li, Phys. Rev. A **81**, 032120 (2010).
 - [25] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
 - [26] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. **62**, 188 (1965).
 - [27] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) **464**, 1301 (2010).
 - [28] J. I. Cirac, M. Lewenstein, K. Molmer, and P. Zoller, Phys. Rev. A **57**, 1208 (1998).
 - [29] S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004).
 - [30] F. Leyvraz and W. D. Heiss, Phys. Rev. Lett. **95**, 050402 (2005).
 - [31] W. H. Zurek, Phys. Rev. A **67**, 012320 (2003).
 - [32] X. Wang and K. Mølmer, Eur. Phys. J. D. **18**, 385 (2002).
 - [33] C. Emary and T. Brandes, Phys. Rev. E **67**, 066203 (2003); Phys. Rev. Lett. **90**, 044101 (2003).
 - [34] Q. H. Chen, Y. Y. Zhang, T. Liu, and K. L. Wang, Phys. Rev. A **78**, 051801(R) (2008).
 - [35] M. A. Bastarrachea-Magnani and J. G. Hirsch, arXiv:1108.0703 (2011).
 - [36] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92** 073602 (2004).
 - [37] S. Dusuel and J. Vidal, Phys. Rev. B **71**, 224420 (2005).